A MODIFIED ISENTHALPIC MODEL OF CAVITATION IN PLANE JOURNAL BEARINGS

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Abstract
This paper presents the development of a quasi-homogeneous isenthalpic cavitation flow model, suitably modified to account for thermal cavitation, and its application to the study of plane journal bearings with constant eccentricity. The proposed model treats the cavitating and noncavitating portions of the fluid in a unified manner with the aim of avoiding the use of matching conditions at the phase interface, whose accuracy is questionable in the presence of significant inertial and/or unsteady effects. A non-linear analysis which accounts for the inertia of the lubricant is used to determine the reaction forces caused by the shaft eccentricity both in the viscosity-dominated regime and at intermediate values of the Reynolds number, where the inertia of the lubricant is no longer negligible. The classical iteration method for the Reynolds lubrication equation (Muster and Sternlicht, 1965; Mori and Mori, 1991; Reinhardt and Lund, 1975) has been extended to the two-phase flow case in order to account for flow acceleration effects in the presence of cavitation. Comparison with available experimental data are shown in a number of representative cases, in order to illustrate the validity and the capabilities of the proposed model for the analysis of cavitating flows in journal bearings, in view of its extension to the case of whirling loads and eccentricities.

1 Introduction
Cavitation or ventilation of the lubricating fluid are frequently experienced in journal bearings and squeeze-film dampers, where they significantly reduce the magnitude of bearing forces and modify their orientation with respect to the fully-wetted flow case (Dowson and Taylor, 1975; Swales, 1975). Future high power density rocket engine turbopumps with propellant lubricated hydrodynamic bearings are particularly critical in this respect because the relatively high saturation pressure and low viscosity of most propellants promote bearing cavitation and operation at larger eccentricity and Reynolds numbers, where the inertia of the lubricant and thermal cavitation effects become significant, enhancing the danger of rotordynamic instabilities. In particular, the correct evaluation of the inertial effects is of special importance in the presence of whirling eccentricities, because even relatively minor changes of the load orientation can significantly modify the rotordynamic damping of the bearing and have crucial consequences on the dynamic stability of the suspended rotor, especially in supercritical applications.

A wide body of literature exists on steady cavitation in bearings, where it is usually treated by properly matching the separate solutions for the liquid and two-phase flow regions (Coyne and Elrod, 1970a, 1970b; Floberg, 1957a, 1957b; Dowson e Taylor, 1979), or using heuristic cavitation algorithms (Elrod and Adams, 1975; Elrod, 1981; Brewe, 1986). More elaborate analyses (Zeidan and Vance, 1990; Feng and Hahn, 1985) make use of experimental or simplified theoretical correlations between the gas and liquid densities, which do not include inertial and thermal cavitation effects in the lubricant. In a comparative study Mori and Mori (1991) convincingly showed that the traditional methods used for locating and matching the cavitation region to the fully wetted flow in bearings yields rather inaccurate results in the prediction of the magnitude and orientation of the bearing load under cavitating conditions. This difficulty is clearly exacerbated by the occurrence of inertial effects of the lubricant, when the extension of the methods developed for operation at low Reynolds numbers is even more uncertain and in many cases unjustified. These considerations suggest that better results should be obtained from those models where phase transitions are the natural consequences of the constitutive behavior of the lubricant.

A unified treatment of the cavitating and noncavitating regions of the flow is especially preferable for the analysis of cavitation in whirling bearings with cryogenic lubricants, where the operation at high Reynolds numbers under unsteady conditions makes the applicability of "ad hoc" assumptions on the location and behavior of the phase interface even more questionable. The present analysis specifically aims at the development and validation of a physically-based unified model of the single-phase/two-phase flow in journal bearings capable of accounting for inertial and thermal cavitation/ventilation effects, in view of its future application to the study of unsteady rotordynamic effects in lubricated bearings and squeezed film dampers. Previous analyses (d'Auria, d'Agostino and Brennen 1996; d'Agostino 1999) showed that bubble dynamic effects usually have a negligible impact on the bearing reaction forces. The two-phase flow essentially responds in quasi-static fashion to external pressure changes and therefore in the present analysis is studied using a barotropic approximation, suitably modified to account for thermal cavitation effects. Despite the intrinsic limitations of this approach and the simplifying assumptions introduced to obtain a workable model, we are confident that the present analysis clearly identifies the main phenomena and parameters involved and provides useful indications on their influence on the fluid dynamic behavior of bearings and squeeze-film dampers operating with dispersed cavitation and/or ventilation.

2 Two-Phase Flow Models

Adiabatic force-free flows of cavitating liquids are practically isenthalpic and, in the absence of inertial and surface tension effects, are characterized by a barotropic relation between the local pressure, p, (equal for both phases) and density, $r = m/V$, of the fluid mixture. This represents a significant advantage of the isenthalpic formulation, as it reduces the order of the differential problem governing the flow field. In order to account for thermal cavitation effects, dominant in cryogenic fluids, one can assume that thermodynamic equilibrium is only established with fraction e_L and e_V of the total volume of the liquid and vapor phases:

$$V = (1-e_L)V_L + e_L V_L + (1-e_V)V_V + e_V V_V$$

while the remaining portions of the two phases behave isentropically (Brennen, 1995). Under these assumptions, using the definitions of the phase densities, $r_L = m_L/V_L$, $r_V = m_V/V_V$, and void fractions, $a_V = V_V/V$, $a_L = V_L/V$, the speed of sound of a mixture of constant mass $m = m_L + m_V$ is:

$$\frac{1}{r c^2} = \frac{1}{r}\frac{dr}{dp} = -\frac{1}{V}\frac{dV}{dp} =$$

$$= e_L \frac{a_L}{r_L}\left(\frac{\P r_L}{\P p}\right)_E + e_V \frac{a_V}{r_V}\left(\frac{\P r_V}{\P p}\right)_E + (1-e_L)\frac{a_L}{r_L}\left(\frac{\P r_L}{\P p}\right)_S + (1-e_V)\frac{a_V}{r_V}\left(\frac{\P r_V}{\P p}\right)_S + \frac{dm_V}{Vdp}\left(\frac{1}{r_L} - \frac{1}{r_V}\right)$$

where $dm_V = -dm_L$ is the mass interaction of the two phases and the indexes E and S indicate differentiation at equilibrium and isentropic conditions, respectively. By evaluating dm_V from the condition of isentropic transformations of the mixture $d(m_L s_L + m_V s_V) = 0$ and using $dh = Tds + dp/r$ to express the derivatives in terms of the enthalpy, the following expression for c is obtained:

$$\frac{1}{r c^2} = \frac{1}{r}\frac{dr}{dp} = \frac{a}{p}\left[(1-e_V)\, f_V + e_V g_V\right] + \frac{1-a}{p}\left[(1-e_L)\, f_L + e_L g_L\right]$$

where $a = a_V = 1 - a_L$ is the void fraction and for each phase $f_V = 1/g_V$, $f_L = p/r_L c_L^2$, g_V and g_L are known functions of the pressure p or the temperature T. Far from the critical pressure p_c, for a number of fluids of technical interest (like water and most liquid propellants) g_L can be approximated by a power law $g_L \cong g^*(p_c/p)^h$, while both f_V and $g_V \cong 0.9$ are of order unity, so that $k_V = (1-e_V)\, f_V + e_V g_V \cong 1$ is nearly constant and the actual value of e_V is practically unimportant. On the other hand, the value of e_L clearly depends on the dispersion of the gaseous phase in the mixture and therefore indirectly on the void fraction a, tending to zero in the fully wetted limit ($a \to 0$) and to unity for high void fractions ($a \to 1$). In the assumption of spherical cavities of radius R, average semi-separation $b = a^{-1/3}R >> R$, and thermal boundary layer thickness d_T (in the liquid), e_L can be estimated to be:

$$e_L \cong \frac{(R+d_T)^3 - R^3}{b^3 - R^3} \cong \frac{a}{1-a}\left[\left(1+\frac{d_T}{R}\right)^3 - 1\right]$$

In the thermally-controlled growth of a cavitating spherical bubble $d_T \cong \sqrt{a_{TL} t}$ and $R \cong R^* \sqrt{t}$, where a_{TL} is the thermal diffusivity of the liquid. Therefore d_T/R is approximately constant and can be considered as a free flow parameter, which determines the value of e_L and accounts for the influence of thermal cavitation effects. Clearly d_T/R ranges from zero in frozen flows to $d_T/R >> 1$ in isothermal equilibrium flows. Conversely, for any given value of d_T/R, the above expression of e_L is valid for void fractions not exceeding the critical value a_e corresponding to $e_L = 1$.

The value of $d_T/R \cong \sqrt{a_{TL}}/R^*$ can also be estimated from the expression of:

$$R^* = \frac{\left(-C_{p_{\min}} - s\right)U_\infty^2}{\Sigma(T_L)}$$

(Brennen 1995) as a function of the cavitation number $s = \left(p_\infty - p_V\right)/\frac{1}{2} r_L U_\infty^2$, the reference flow velocity U_∞, the local pressure coefficient $C_p \cong C_{p_{\min}}$, and the thermodynamic parameter $\Sigma(T_L) = r_{\text{sat}}^2 Q_{\text{vap}}^2 / r_L^2 c_{pL} T_L a_{TL}^{1/2}$, which depends on the latent heat of vaporization Q_{vap} and the saturation conditions of each fluid and is a strong function of its temperature T_L. However, the value of R^* for a single bubble is clearly overestimated, since the presence of multiple bubbles reduces their growth rate. From the combined-phase continuity and momentum equations of the mixture it follows that two cavitating flows (indexes 1 and 2) with the same boundary conditions have equal velocity and pressure fields if $a_1 = a_2$. Therefore, since $nR^3 \cong a/r_L(1-a)$ where n is the number of active cavitation nuclei per unit mass of the mixture, it follows that:

$$\frac{R_2^*}{R_1^*} = \frac{R_2}{R_1} = \left[\frac{n_1 a_2 (1-a_1)}{n_2 a_1 (1-a_2)}\right]^{1/3} = \left(\frac{n_1}{n_2}\right)^{1/3}$$

For the flow with a single bubble $n_1 \cong 1/r_L V_C$, where V_C is the cavitation volume ($p \le p_V$). Corrected values of $d_T/R \cong \sqrt{a_{TL}}/R_2^*$ for cavitation in waters near room conditions with typical values of the concentrations of active nuclei (10 to $100\ \text{cm}^{-3}$) are consistent with the estimates of d_T/R obtained by fitting numerical simulations of the cavitating flow with the available experimental data. Extrapolation to higher temperatures is more uncertain, presumably because of the temperature dependence of the active nuclei concentration.

With previous results and the approximation $r = r_L(1-a)$ the equation for the sound speed can be sequentially integrated in closed on $0 \le a(p) \le a_e$ ($e_L(a) \le 1$) and $a_e \le a(p) \le 1$ ($e_L = 1$) with initial condition $a(p_{\text{sat}}) = 0$. The corresponding sound speeds and barotropic relations for water at 20 °C and 70 °C are shown in Figures 1 and 2 for several values of d_T/R. Notice that the equilibrium flow model ($d_T/R \to \infty$) predicts a discontinuity of the sound speed and a corner point in the barotropic curve at $a = 0$ as a consequence of the unphysical assumption $e_L = 1$ also in the limit for $a \to 0$. For finite values of d_T/R down to $d_T/R \to 0$ (frozen flow model) the transition of the density from the vapor to the liquid values becomes more gradual and occurs at lower pressures, while the sound speed increases. In all cases, however, for most values of the void fraction the sound speed c is only a few meters per second, and even lower at room temperature. Hence the flow Mach number of cavitating flows is usually much greater than sonic, with sharp transitions near the boundaries of the cavitation region.

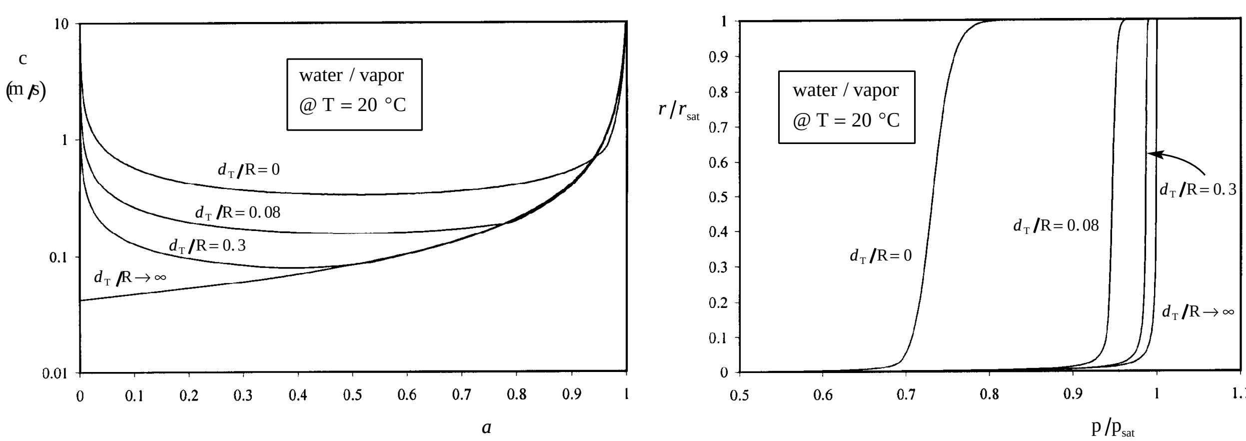

Figure 1. Sonic velocity as a function of the void fraction a and barotropic curve as a function of the pressure p/p_{sat} for water/vapor mixtures at 20 °C and several values of $d_T/R = 0$ (frozen flow model), 0.08, 0.3 and $d_T/R \to \infty$ (equilibrium flow model).

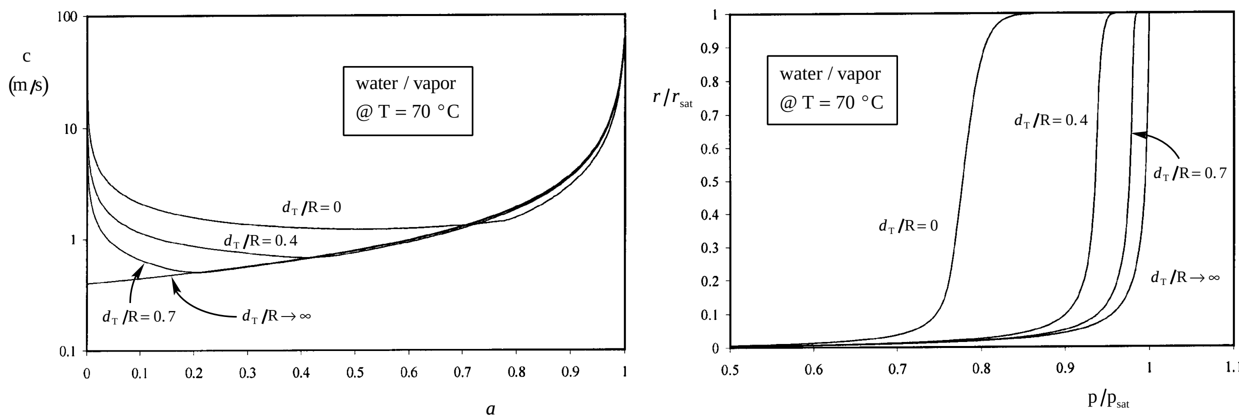

Figure 2. Sonic velocity as a function of the void fraction α and barotropic curve as a function of the pressure p/p_{sat} for water/vapor mixtures at 70 °C and several values of $\delta_T/R = 0$ (frozen flow model), 0.4, 0.7 and $\delta_T/R \to \infty$ (equilibrium flow model).

In the presence of significant mass fractions of noncondensable gas in the cavities, thermal effects are greatly reduced and a simplified liquid/gas/vapor model can be readily obtained by assuming constant mixture temperature T, constant mass fraction of the noncondensable gas $X_G = m_G/(m_L + m_G + m_V)$, constant vapor pressure $p_V = p_{\mathrm{sat}}(T)$, and perfect gas behavior of the cavity contents. An example of the barotropic relation of water/air/vapor mixtures, derived and integrated with the same procedure used for liquid/vapor systems, is illustrated in Figure 3.

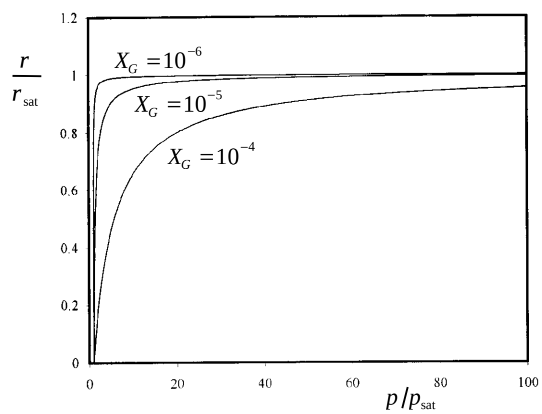

Figure 3. Barotropic curves of water/air/vapor mixtures at room temperature as a function of p/p_{sat} for several values of the gas mass fraction X_G at room conditions.

3 Two-Phase Bearing Flow

In the absence of significant inertial effects, the laminar compressible flow in a plane journal bearing supporting a shaft of radius R, radial clearance C, eccentricity e, and rotational velocity ω (see Figure 4) is held by the classical Reynolds lubrication equation (Pinkus e Sternlicht, 1961; Szeri, 1998) in the form:

$$\frac{d}{d\theta}\left(\rho H^3 \frac{dp}{d\theta}\right) = 6\mu\omega R^2 \frac{d}{d\theta}(\rho H)$$

where $\Lambda = 6\mu\omega R^2$ is the bearing number, $H \cong C - e\cos\theta$ is the height of the lubricating film, and the fluid density is obtained for each value of the pressure from the barotropic relation of the isenthalpic two-phase flow model. The above equation must be solved with periodic boundary conditions, $p|_\theta = p|_{\theta+2\pi}$, $dp/d\theta|_\theta = dp/d\theta|_{\theta+2\pi}$. In addition, in infinitely long bearings either the supply pressure of the lubricant, p^*, is specified at the appropriate location θ^*, or the overall void fraction of the fluid film is assigned:

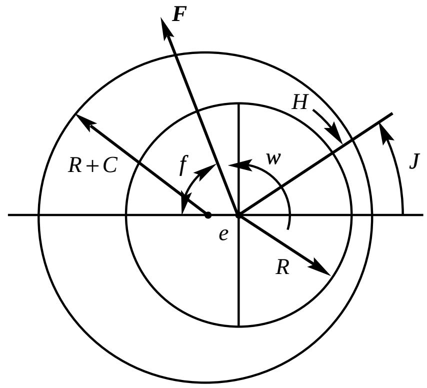

Figure 4. Schematic of the bearing flow configuration and forces.

$$\bar{a} = \frac{1}{2p\,RC}\int_{J}^{J+2p} a\,HR\,dJ$$

In high power density turbomachines the increase of the shaft speed, possibly combined with the use of low-viscosity lubricants, can raise the modified Reynolds number $\mathrm{Re}^* = wC^2/n_L$ of the bearing significantly above unity. In this case the inertial terms of the momentum equation for the lubricating fluid should be taken into account and the Reynolds equation in the above form is no longer accurate. In the present work the inertial effects have been included using the iteration method (Kahlert, 1947), assuming that they represent a small correction to the Reynolds solution (index 0) and can be computed using the velocity field $\boldsymbol{u}_0$ obtained in the absence of the inertial forces. With this approximation, the differential problem for the corrected flow variables (index 1) writes:

$$\frac{dp_1}{dJ} = \frac{\Lambda}{H^2} - \frac{2\Lambda\dot{m}_1}{r_1 wRH^3} + \frac{r_0 w^2 R^2}{\Lambda^2}\left[\frac{\Lambda^2}{5} + \frac{27}{70}H^4\left(\frac{dp_0}{dJ}\right)^2 - \frac{51}{70}\Lambda H^2\frac{dp_0}{dJ}\right]\left(\frac{d\ln H}{dJ} + \frac{1}{r_0 c_0^2}\frac{dp_0}{dJ}\right)$$

where $\dot{m}_1$ is the corrected mass flow rate across the lubricating film.

A single shooting method with fifth order Runge-Kutta integration and self-adaptive step size (Press *et al.*, 1992) has been chosen to numerically integrate the two point boundary value problem for the Reynolds equation, with or without corrections for inertial effects depending on the value of the modified Reynolds number $\mathrm{Re}^* = wC^2/n_L$. A modified multidimensional Newton-Raphson method (Stoer, 1974) has been used to iterate on the unknown initial conditions until the end-point boundary conditions are met with the required accuracy. The convergence and speed of the method are usually excellent, but appreciably degrade in flows with extensive cavitation and low temperature (negligible vapor pressure), especially at high Reynolds numbers and relative eccentricities.

4 Results and Discussion

The two-phase flow model and the numerical scheme for the prediction of cavitation in journal bearings have been validated against the experimental results by Floberg (1957a). Figure 5 shows, for instance, the comparison of the pressure measurements (dots) with the predicted pressure distribution (solid line) in a long journal bearing ($L/D >> 1$) with extensive cavitation. The agreement with the experimental data is quite satisfactory. More importantly for the validation of the proposed two-phase flow model, both the extent and shape of the pressure distribution in the cavitating region are accurately predicted by the isenthalpic approximation without the need for “ad hoc” assumptions. Besides, the flow Mach number M based on the mass-averaged velocity of the lubricating film transitions from negligible values in the fully wetted liquid to highly supersonic conditions ($M \cong 2 \div 3$) in the cavitating region, demonstrating the robustness and stability of the numerical scheme. In all cases where a comparison was possible, the computed values of the bearing force $\boldsymbol{F}$ and attitude angle f are within a few percent of Floberg’s experimental results. Figure 6 illustrates the effect of ventilation by comparing the pressure profiles predicted by the LV and LGV models for the same value of the average void fraction $\bar{a}$ and for several values of the gas mass fraction X_G in the two-phase mixture. The presence of the noncondensable gas produces a general increase

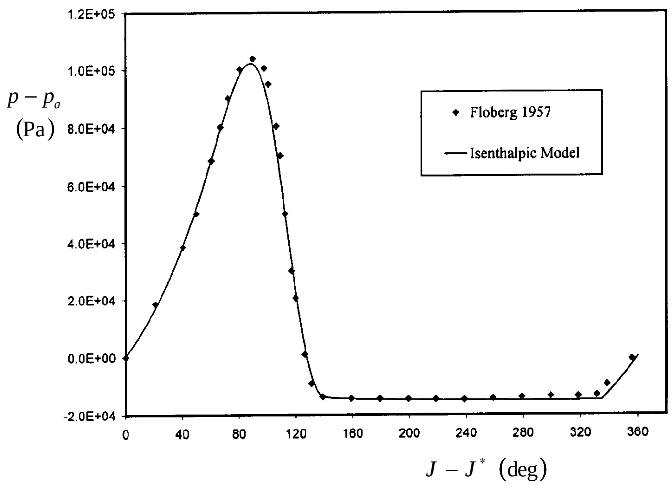

Figure 5. Comparison of the relative pressure measurements by Floberg, 1957a (dots), with the azimuthal pressure distribution predicted by the isenthalpic model (solid line) in a long journal bearing ($L/D >> 1$) with extensive cavitation. The clearance ratio is $C/R = 8.25 \cdot 10^{-3}$, the relative eccentricity is $e = e/C = 0.69$, the modified Reynolds number is $\mathrm{Re}^* = wC^2/n_L = 0.6$, the vapor pressure of the lubricant is $p_{\mathrm{sat}} \cong 85.4$ kPa, and the bearing number is $\Lambda = 0.0155$ N.

of the pressure distribution. The low-pressure region is reduced, the pressure is prevented from dropping below the saturation value and its profile becomes less flat than in the presence of vaporous cavitation.

When the modified Reynolds number of the bearing grows larger than unity, the inertial effects of the lubricant start to appreciably modify the viscous solution of the Reynolds' equation. Their influence is well illustrated in Figure 7, relative to a bearing operating at different Reynolds numbers and the same value of the total void fraction $\bar{a} = 0.154$. The solution in the absence of inertial effects ($\mathrm{Re}^* << 1$) is also shown for comparison. The pressure distributions of Figure 7 indicate that the recompression of the fluid film is significantly more pronounced at higher Reynolds numbers and that the pressure reaches its maximum slightly earlier than in the absence of inertial effects. These results are consistent with the observed trends in bearings at high rotational speeds (see for instance Burton and Hsu, 1974, and Roberts and Hinton, 1982), thus providing an indirect validation of the flow model also at intermediate values of the Reynolds number.

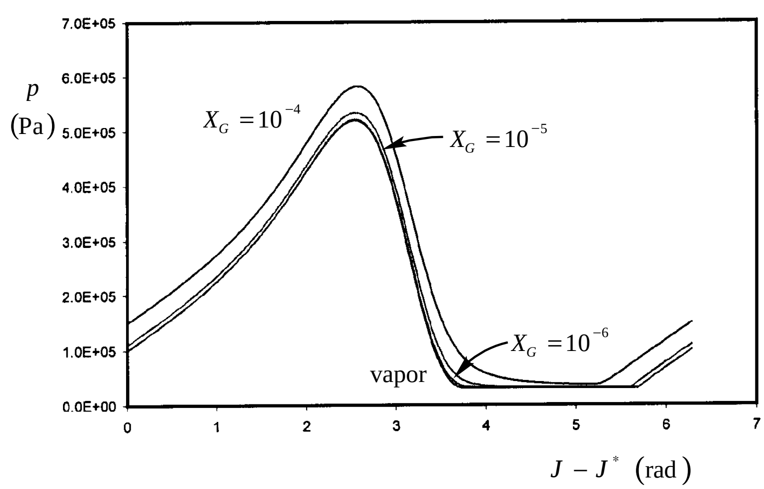

Figure 6. Comparison of the azimuthal pressure distributions predicted by the LV and LGV models for $\bar{a} = 0.154$ and several values of the gas mass fraction in the mixture $X_G = 10^{-4}$, 10^{-5} and 10^{-6}. The clearance ratio is $C/R = 2.9 \cdot 10^{-3}$, the relative eccentricity is $e = e/C = 0.6$, the modified Reynolds number is $\mathrm{Re}^* = wC^2/n_L = 0.08$, the vapor pressure of the lubricant is $p_{\mathrm{sat}} \cong 35$ kPa, and the bearing number is $\Lambda = 0.009$ N.

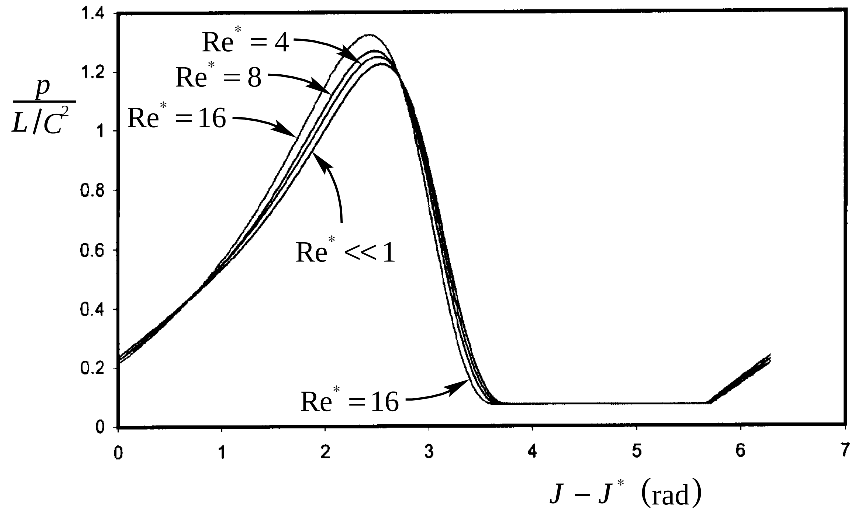

Figure 7. Comparison of the nondimensional azimuthal pressure distributions predicted by the isenthalpic LV model for $\bar{a} = 0.154$ and several values of the modified Reynolds number $\mathrm{Re}^* = wC^2/n_L = 0$, 4, 8 and 16. The clearance ratio is $C/R = 2.91 \cdot 10^{-3}$, the relative eccentricity is $e = e/C = 0.6$, the vapor pressure of the lubricant is $p_{\mathrm{sat}} \cong 35$ kPa, and the bearing number is $\Lambda = 0.009$ N.

The magnitude and orientation of the bearing load for the same values of the flow parameters are shown in Figure 8. The results indicate that inertial effects only induce a moderate increase of the bearing force, but significantly modify the attitude angle. In particular, the influence of inertial effects on the orientation of the bearing force is especially manifest at low and intermediate values of the relative eccentricity. On the other hand, in the limit for high eccentricities the attitude angle tends to become essentially independent on the Reynolds number. The correct evaluation of the load orientation is of special importance in whirling shafts, which we plan to consider in more detail in a later publication. In this case even relatively minor changes of the attitude angle can significantly modify the rotordynamic damping of the bearing. This, in turn, can have important consequences on the dynamic stability of the suspended rotor, especially in supercritical applications, where transient operation near the resonance conditions cannot be avoided.

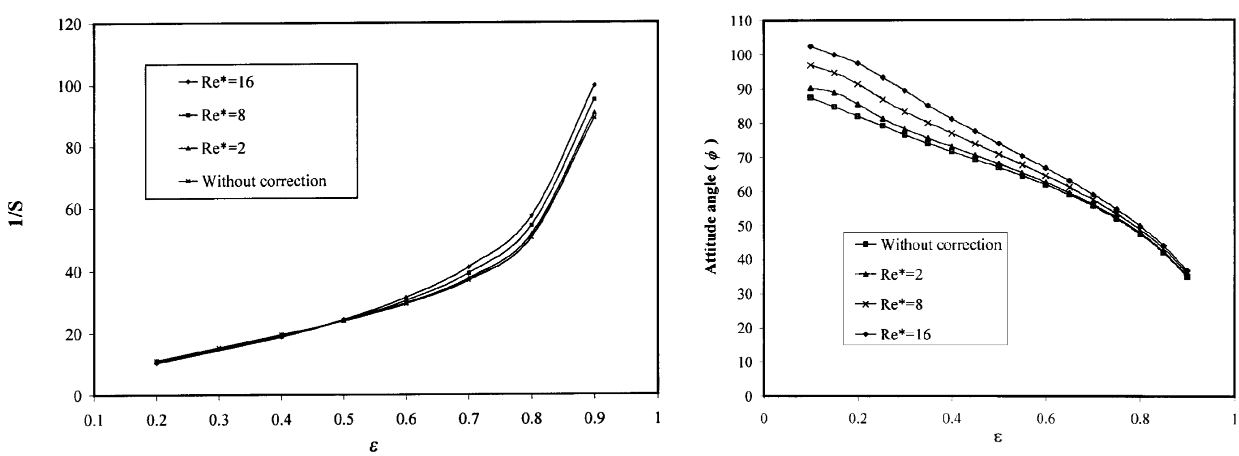

Figure 8. Comparison of the nondimensional load $1/S$ and attitude angle f predicted by the isenthalpic LV model as a function of the relative eccentricity e for $\bar{a} = 0.154$ and several values of the modified Reynolds number $\mathrm{Re}^* = wC^2/n_L = 0$, 2, 8 and 16. The vapor pressure of the lubricant is $p_{\mathrm{sat}} \cong 35$ kPa.

5 Conclusions

Pertinent data from plane journal bearings operating under cavitation/ventilation conditions at low values of the modified Reynolds numbers compare quite well with the bearing pressure distribution, the extent of the two-phase flow region, and the reaction force intensity and orientation predicted by the modified isenthalpic cavitation/ventilation models. The close agreement with the experimental results confirms the validity of the approach used in the derivation of the proposed models and clearly indicates that they are indeed capable of dealing in a unified manner with the two-phase flow phenomena in the lubricating film of journal bearings, without the need of uncertain assumptions or semi-empirical criteria for locating and matching the phase interfaces. Quantitative validation of the model predictions for cavitating bearings at intermediate values of the modified Reynolds number has not been possible due to the lack of reference data in the literature, but the results obtained in this case for the effect of the inertia of the lubricant on the pressure distribution are in agreement with the observed trends in superlaminar bearings at moderate values of $\mathrm{Re}^* > 1$. The successful validation of the proposed models strongly supports the possibility of usefully extending their application to the analysis of cavitating bearings with unsteady loads and eccentricities. In spite of the simplifying assumptions introduced to generate workable models, the present analysis clearly highlights the potential importance of two-phase flow effects in the dynamics of journal bearings and provides a useful introduction to the study of the rotordynamic characteristics and forces in whirling bearings and squeezed-film dampers operating under unsteady cavitation and/or ventilation conditions.

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